The Lorenz Manifold: Crochet and Curvature

Hinke M Osinga and Bernd Krauskopf
Bristol Centre for Applied Nonlinear Mathematics
Department of Engineering Mathematics
University of Bristol, Bristol BS8 1TR, UK
E-mail: H.M.Osinga@bristol.ac.uk, B.Krauskopf@bristol.ac.uk

Abstract

We present a crocheted model of an intriguing two-dimensional surface — known as the Lorenz manifold — which illustrates chaotic dynamics in the well-known Lorenz system. The crochet instructions are the result of specialized computer software developed by us to compute so-called stable and unstable manifolds. The implicitly defined Lorenz manifold is not only key to understanding chaotic dynamics, but also emerges as an inherently artistic object.

1 Introduction

Many people know a version of the saying that a butterfly flapping its wings in Brazil can cause a tornado in Texas. This is also referred to as the *butterfly effect*, and it was first introduced by Edward Lorenz in 1963 [8] to illustrate extreme sensitivity of the weather. If a small effect such as the wing flap of a butterfly can be responsible for creating a large-scale phenomenon like a tornado, then it is impossible to predict the behaviour of a complex system. Of course, there are many butterflies that may or may not flap their wings...

Our work concentrates on the fact that one can still extract information from such a complex system and make predicitions at a more qualitative level via the computation of so-called *invariant manifolds*. The unpredictability of a chaotic system can be translated into the complexity of the geometry of these manifolds, which are two-dimensional surfaces in three-dimensional space in many cases. These surfaces emerge as inherently artistic shapes that are already implicitly contained in the mathematical model. We have developed computational methods to find and visualize such manifolds. Even better, our method can be interpreted as crochet instructions. This allows us to visualise the chaotic dynamics with a real-life crocheted model.

2 The Lorenz system

The classic model studied by Lorenz when he discovered the butterfly effect was a simplified set of seven equations describing the rising and cooling of hot air (thermal convection) in the atmosphere. He later managed to create the same dynamical effect in an even simpler model that is now known as the Lorenz system:

$$\left\{\begin{array}{rcl} \dot{x} &=& \sigma(y-x), \\ \dot{y} &=& \varrho x - y - xz, \\ \dot{z} &=& xy - \beta z. \end{array}\right. \tag{1}$$

The classic values of the parameters, as introduced by Lorenz, are $\sigma = 10$, $\varrho = 28$, and $\beta = 2\frac{2}{3}$. The system is given in the form of a set of three ordinary differential equations where the vector

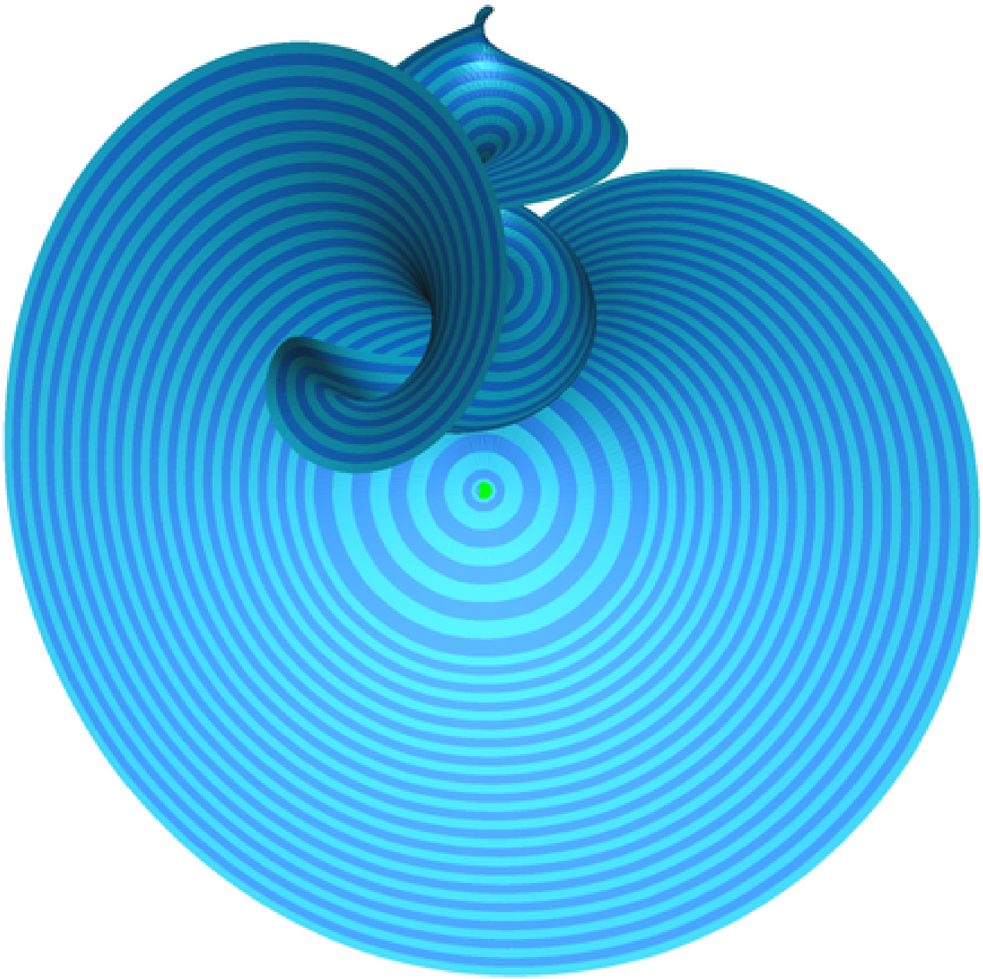

Figure 1: *The Lorenz manifold as computed by our algorithm.*

$(\dot{x}, \dot{y}, \dot{z})$ represents the (instantaneous) velocity of a particle at position (x, y, z). The Lorenz system is deterministic, which means that for each particle position Eqs. (1) uniquely describe the future and past of the dynamics. Remarkably, this is not enough to make predicitions for even relatively short time scales; we refer to [2] for a popular account and to [12] for more details on deterministic systems.

An important feature of system (1) is its symmetry: the behaviour of a particle starting at (x, y, z) is essentially the same as that of a particle starting at $(-x, -y, z)$. That is, any solution path in space can be transformed into another solution path by rotating it by 180° about the z-axis. The z-axis itself is invariant under the dynamics, which means that a particle on the z-axis will stay on it.

3 The Lorenz manifold

The origin of the Lorenz system (1) is an equilibrium, that is, the velocity vector at $(x, y, z) = (0, 0, 0)$ is the zero vector. It is a saddle point with a two-dimensional stable manifold, also called the Lorenz manifold. This manifold consists of all points that approach the origin in forward time. The chaotic dynamics of the system is essentially organised by how particles that pass close to

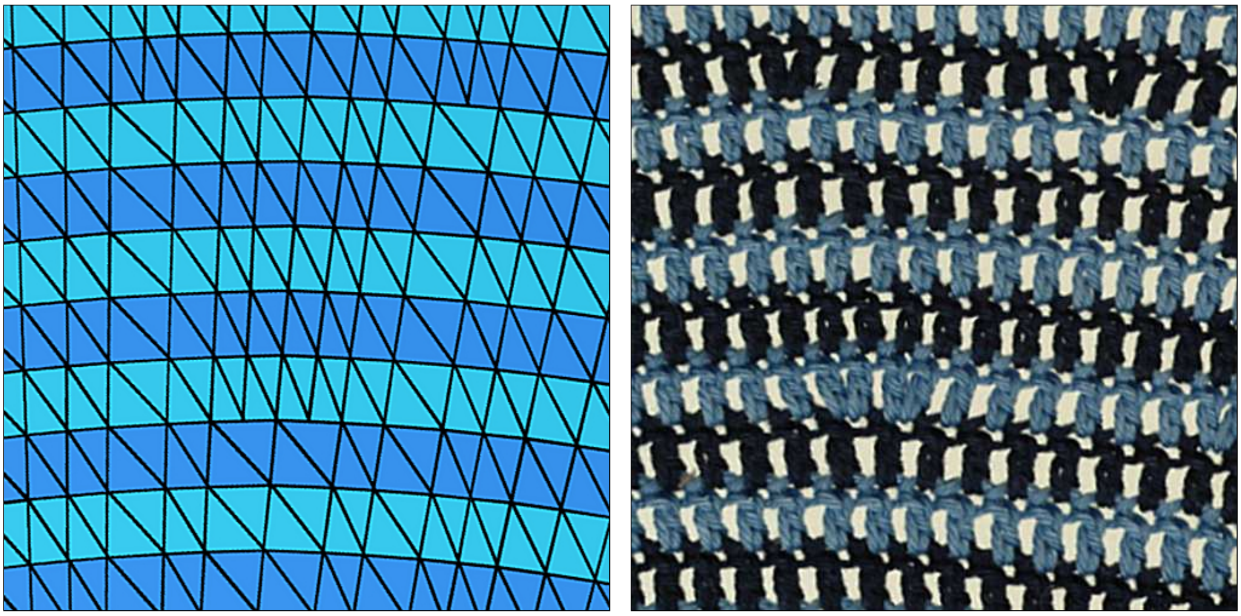

Figure 2: *The computed and corresponding crocheted mesh of part of the Lorenz manifold.*

the origin get pushed away again. Namely, the Lorenz manifold is the two-dimensional boundary surface that separates points that locally get pushed towards negative x-values from those that are locally pushed in the positive x-direction.

It has long been an open problem to find and visualize the Lorenz manifold, because there is no explicit formula for it. The first sketches of the Lorenz manifold appeared in 1982 in [1], but computational methods have been developed only quite recently; see [7] for an overview. The method we developed grows the surface as a set of concentric rings, starting from a small circle about the origin, in such a way that the surface grows with equal speed in all radial directions until a prescribed (geodesic) distance is reached. Details can be found in [4, 5], where we discuss the accuracy of the computations, while [9, 6] provide a less technical explanation. Figure 1 shows the Lorenz manifold computed with our method up to geodesic distance 110.75.

Our method represents the manifold as a triangulation between computed mesh points. At each step a new ring of mesh points and triangles is added. The images in this paper highlight consecutive rings by alternating light and dark blue. As the manifold grows, mesh points are added (or removed) to ensure an even distribution of mesh points and, hence, a faithful representation of the mathematical object.

This systematic way of building up the mesh can be interpreted directly as crochet instructions, which we published in [10]. Our crocheted model is built up round by round of crochet stitches of the appropriate length, where stitches are added or removed as dictated by the algorithm. Figure 2 shows a direct comparison between a part of the computed and the corresponding crocheted mesh. Notice in particular where mesh points and, consequently, crochet stitches have been added.

The result of crocheting the entire Lorenz manifold (up to geodesic distance 110.75) is shown in Figure 3. It is a floppy object that is impossible to lay down flat on a table. While the lower part is actually almost flat, the upper part (positive z-values) ripples considerably. This is due to the negative curvature of this part of the surface (compare with Figure 1), which is locally encoded simply by the way stitches are added during the crochet process. Negative curvature is generated during a growth process by a faster growth in the lateral direction. This can be found in nature, for example, in curly-leaf lettuce. The principle can also be used to crochet examples of hyperbolic

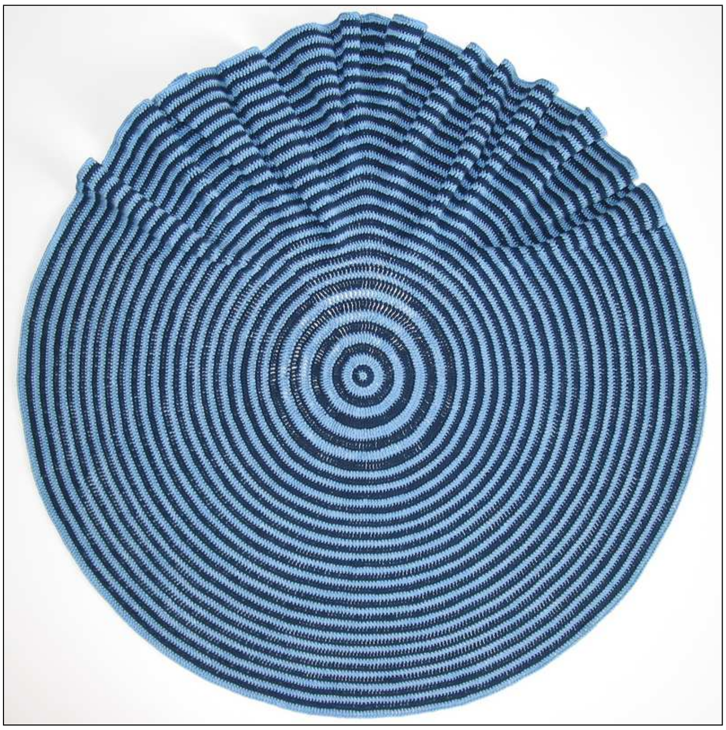

Figure 3: *The crocheted Lorenz manifold before mounting.*

planes [3], which are characterized by constant negative curvature. The Lorenz manifold, on the other hand, is not a (hyperbolic) plane, rather its curvature varies from point to point on the surface.

The model of the Lorenz manifold is obtained from the floppy crocheted object by mounting it with a stiff z-axis, a rim wire, and two additional supporting wires; precise mounting instructions can be found in [10]. Figure 4 shows the result seen from approximately the same viewpoint as in Figure 1. The mesh structure is brought out by the white background. Figure 5 shows a close-up view of the crocheted Lorenz manifold near the z-axis, where a black background is used to emphasize the surface geometry. In this region the local curvature is maximal, which creates the helical structure after mounting.

4 Mathematics or art?

Our motivation for creating the crocheted model of the Lorenz manifold was to have a three-dimensional hands-on model of this intriguing surface. However, apart from simply appealing to the specialists, the crocheted object is an excellent tool to communicate the beauty of complicated mathematical objects and ideas. This found resonance with the general public who perceived our creation as a piece of mathematical art. What is more, many people followed our published crochet and mounting instructions to produce their own Lorenz manifold. More information and photographs can be found on our dedicated website [11].

Stable and unstable manifolds of chaotic systems have complex and beautiful geometry. However, they are ‘hidden’ in the governing mathematical equations and must be brought to light with specialised algorithms. We hope that the example of the Lorenz manifold may serve as an inspiration to artists.

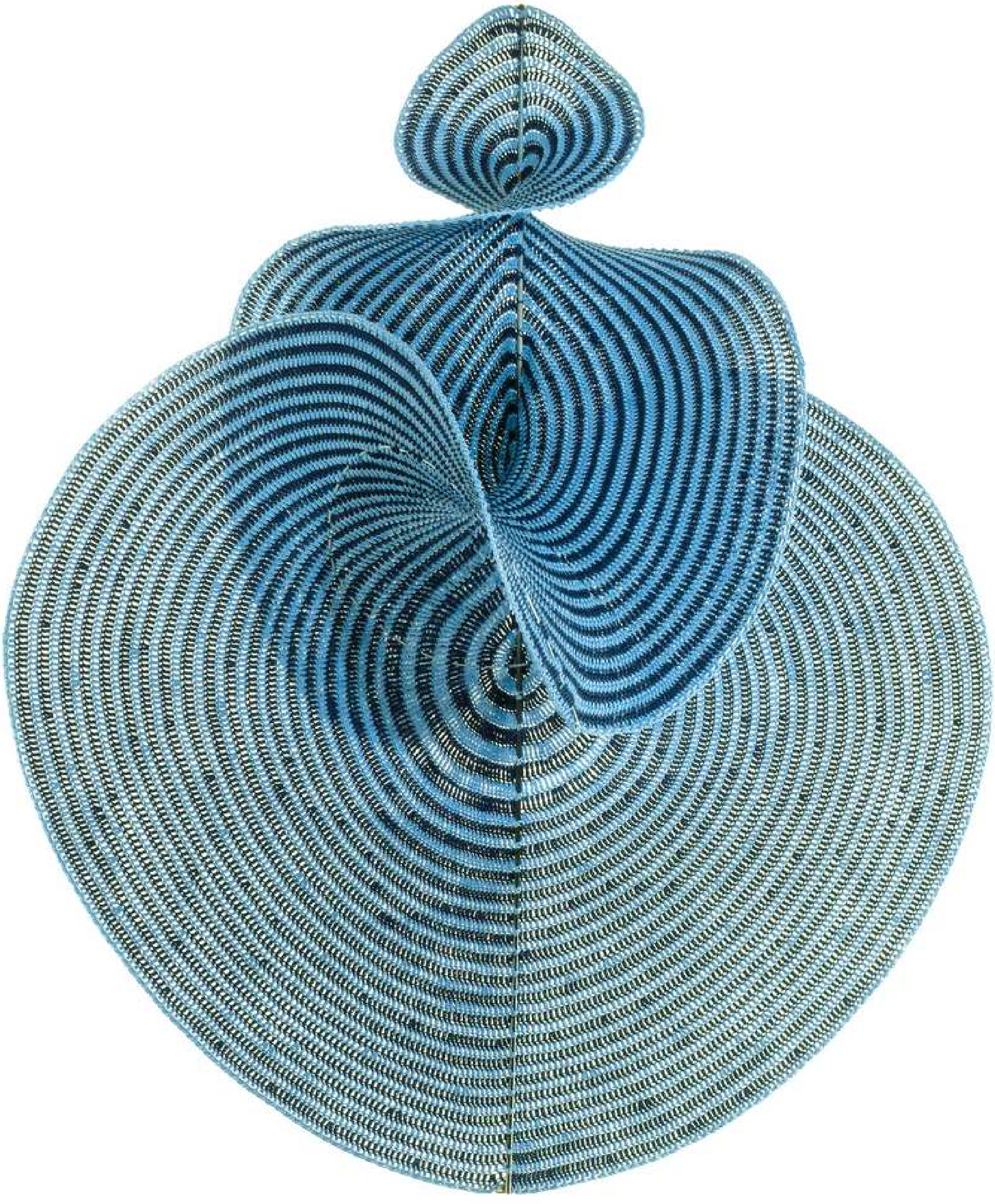

Figure 4: *The crocheted Lorenz manifold after mounting.*

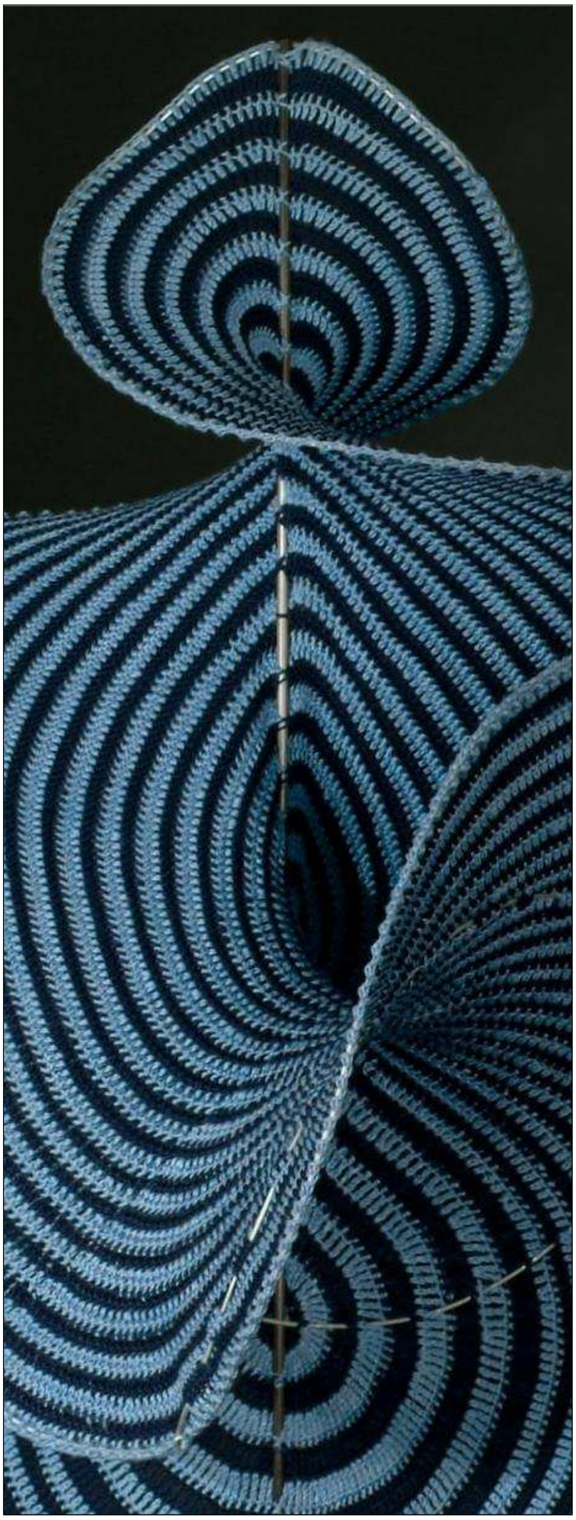

Figure 5: *Close-up of the crocheted Lorenz manifold.*

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